2018

MATHEMATICS

(Major)

Paper : 6.4

(Discrete Mathematics)

Full Marks : 60

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. Answer the following questions : $1\times7=7$

   (a) State well-ordering principle (WOP) of positive integers.

   (b) If a is a non-zero integer, then show that $\gcd(a, 0) = |a|$.

   (c) Find the value of $\phi(180)$.

(d) State Euler's theorem on congruences.

(e) Show that the Diophantine equation $2x+4y=5$ has no solution.

(f) Write a primitive Pythagorean triple of the form 16, y, z.

(g) Give an example of a reduced residue set (r.r.s.) modulo 5.

2. Answer the following questions : 2×4=8

(a) For integers a and b, if $(a, 4)=2$, $(b, 4)=2$, then show that $(a+b, 4)=4$.

(b) Show that $1^2, 2^2, 3^2, \ldots, m^2$ is not a CSR $(\text{mod}\, m)$ if $m>2$.

(c) If x, y, z is a primitive Pythagorean triple, then show that $(x, y)=1$, $(y, z)=1$, $(z, x)=1$.

(d) Find the integers which when divided by 6 and 15 leave remainders 5 and 8 respectively. (Do not use Chinese remainder theorem).

3. Answer the following questions : 5×3=15

(a) Prove that for any $a, b \in \mathbb{Z}$, $b \neq 0$, there are unique integers q and r such that $a = bq + r$, with $0 \leq r < |b|$.

Or

If p_n denotes the nth prime, then show that $p_n \leq 2^{2^{n-1}}$.

(b) State and prove Chinese remainder theorem.

Or

Define Möbius function μ. Show that μ is a multiplicative arithmetic function.

(c) Define a primitive Pythagorean triple. Show that the radius of the inscribed circle of a Pythagorean triangle is always an integer.

Or

Show that an odd prime p can be expressed as a sum of two squares if and only if $p \equiv 1 (\text{mod}\, 4)$.

4. Answer *either (a) or (b)* : 10

(a) *(i)* If $n = p_1^{k_1} p_2^{k_2} \dots p_r^{k_r}$ is the prime decomposition of a positive integer $n > 1$, then show that the positive divisors of n are precisely those integers of the form

$$d = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r},\ 0 \le \alpha_i \le k_i,\ i = 1, 2, \dots, r$$ 5

(ii) Let f be a multiplicative arithmetic function. Then show that $\sum_{d|n} f(d)$ is also a multiplicative arithmetic function. 5

(b) *(i)* Prove that

$$\sum_{d|n} \phi(d) = n$$ 5

(ii) Show that, for $n > 1$

$$\sum_{\substack{(k,\ n)=1 \\ 1 \le k < n}} k = \frac{1}{2} n\phi(n)$$

i.e., the sum of the positive integers less than n and relatively prime to n is $\frac{1}{2} \cdot n \cdot \phi(n)$.

Also show that $\phi(p^\alpha) = p^\alpha\left(1 - \frac{1}{p}\right)$, where p is a prime and α is a positive integer. 2+3=5

5. Answer *either (a) or (b)* : 10

(a) *(i)* Define Boolean algebra and give an example. Show that addition is distributive in a Boolean algebra. 2+3=5

(ii) Write the Boolean expression in x, y, z which takes the value 0 if and only if at least two of the variables take the value 1. 5

(b) *(i)* Prove that every Boolean expression which does not contain any constants can be reduced to a Boolean expression in conjunctive normal form (CNF). 5

(ii) Show that in the algebra of switching circuits, the following law holds : 5

$$x \cdot (y + z) = x \cdot y + x \cdot z$$

6. Answer *either* (a) *or* (b) : 10

(a) (i) Translate the following composite sentence into symbolic notation using statement letters to stand for prime components : 3

If either labour or management is stubborn, then the strike will be settled if and only if government obtains an injunction, but the troops are sent into the mills.

(ii) Define a statement formula. Construct the truth table for the statement formula, $p \to (q \wedge \sim p)$. 2+2=4

(iii) Show that the system $\{\sim, \wedge\}$ is an adequate system of connectives. 3

(b) (i) State and prove the 'principle of substitution' of propositional calculus. 1+3=4

(ii) Prove that the system $\{\wedge, \to\}$ is not an adequate system of connectives. 3

(iii) Define statement bundle. Show that the collection B of all statement bundles is a Boolean algebra. 1+2=3

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