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3 (Sem–6/CBCS) MAT HE 5/6/7

2022

MATHEMATICS

(Honours Elective)

DSE–(H)–4

Answer the Questions from any one Option.

OPTION–E

(Rigid Dynamics)

Paper : MAT–HE–6056

OPTION–F

(Group Theory–II)

Paper : MAT–HE–6066

OPTION–G

(Mathematical Finance)

Paper : MAT–HE–6076

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

Contd.

OPTION–E

(Rigid Dynamics)

Paper : MAT–HE–6056

1. Answer ***any ten*** questions : 1×10=10

(a) Define moment of inertia of a body.

(b) Define equimomental systems.

(c) State D'Alembert's principle.

(d) What are generalized coordinates ?

(e) What is the conservation law of angular momentum of a rigid body ?

(f) Define the momental ellipsoid.

(g) Define angular momentum.

(h) What is the moment of inertia of a circular disc of mass M and radius a about a diameter ?

(i) Define the product of inertia of a rigid body.

(j) State the perpendicular axes theorem on moments of inertia.

(k) Define radius of gyration of the rigid body about a line.

(l) Define the centre of percussion.

(m) Define the principal axes of a rigid body at a point O of the body.

(n) A particle moves on the surface of a sphere. What is the degree of freedom of the particle ?

(o) Define the point of suspension.

2. Answer ***any five*** questions : 2×5=10

(a) Find the moment of inertia of a rod of length $2a$ and mass M about a line through one of its extremities perpendicular to its length.

(b) Prove the perpendicular axes theorem.

(c) A rigid body consists of 3 particles of masses 3 units, 5 units and 2 units located at the point $(-1,0,1),(2,-1,3)$ and $(-2,2,1)$ respectively. Find the moments of inertia about *(i)* the y-axis, and *(ii)* the z-axis.

(d) A rigid body of mass 2 units rotates with angular velocity $\vec{\omega} = (1,1,-1)$ and has the angular momentum $\vec{\Omega} = (2,3,-1)$. Find the kinetic energy of the body.

(e) A particle of mass 3 units located at the point (2,0,0). The particle rotates about O with angular velocity $\vec{\omega} = \hat{k}$. Find the angular momentum of the particle about O.

(f) Find the moment of inertia of a solid circular cylinder of radius a, height h and mass M about the axis of the cylinder.

(g) A rigid body with one point fixed rotates with angular velocity $\vec{\omega}$ and has angular momentum $\vec{\Omega}$. Prove that the kinetic energy is given by $\frac{1}{2}(\vec{\omega}.\vec{\Omega})$.

(h) Find the length of the simple equivalent pendulum for a circular disc when the axis is a tangent to it.

3. Answer ***any four*** questions : 5×4=20

(a) Find the moment of inertia of a hollow sphere of radius a and mass M about a diameter.

(b) If the moments and products of inertia of a body about three perpendicular concurrent axes are known, find the moment of inertia of the body about the line $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$.

(c) Use D'Alembert's principle to obtain the equations of motion of any rigid body.

(d) A rough uniform board of mass m and length $2a$ rests on a smooth horizontal plane and a man of mass M walks on it from one end to the other. Show that the distance through which the board moves in this time is $\frac{2Ma}{M+m}$.

(e) An elliptical lamina is such that when it swings about one latus rectum as a horizontal axis, the other latus rectum passes through the centre of oscillation. Prove that the eccentricity is $\frac{1}{2}$.

(f) Obtain the Lagrangian for a simple pendulum and hence derive the equations of motion of the body.

(g) Show that the moment of inertia of an ellipse of mass M and semi axes a and b about a tangent is $\frac{5}{4}Mp^2$, where p is perpendicular from the centre on the tangent.

(h) Show that the momental ellipsoid at the centre of an elliptic plate is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+z^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)= \text{constant}.$$

4. Answer ***any four*** questions : 10×4=40

(a) Show that the moment of inertia of a right solid cone, whose height is h and the radius of whose base is a, is $\frac{3Ma^2}{20}\frac{6h^2+a^2}{h^2+a^2}$ about a slant side, and $\frac{3M}{80}\left(h^2+4a^2\right)$ about a line through the centre of gravity of the cone perpendicular to its axis.

(b) With usual notation, prove the formula $\frac{1}{2}M\left(V^2+k^2\omega^2\right)$ for the K.E. of a lamina moving in its plane.

(c) Define impressed forces and effective forces. A uniform rod OA of length $2a$, free to turn about its end O revolves with uniform angular velocity ω about the vertical OZ through O and is inclined at a constant angle α to OZ. Show that the value of α is either zero or $\cos^{-1}\left(\frac{3g}{4a\omega^2}\right)$.

(d) A rod of length $2a$ is suspended by a string of length l attracted to one end, if the string and rod revolve about the verticle with uniform angular velocity and their inclination to the verticle be θ and ϕ respectively, show that

$$\frac{3l}{a}=\frac{(4\tan\theta-3\tan\phi)\sin\phi}{(\tan\phi-\tan\theta)\sin\theta}.$$

(e) A pendulum is supported at O and P is the centre of oscillation. Show that, if an additional weight is rigidly attached at P, the period of oscillation is unaltered.

(f) Two equal uniform rods AB and AC, are freely hinged at A and rest in a straight line on a smooth table. A blow is struck at B perpendicular to the rods; show that the kinetic energy generated is $\frac{7}{4}$ times what it would be if the rods were rigidly fastened together at A.

(g) A uniform rod is held in a vertical position with one end resting upon a perfectly rough table, and when released rotates about the end in contact with the table. Find the motion.

(h) Derive Lagrange's equation in generalized coordinates.

(i) Derive the equations of motion of a rigid body in two dimensions when the forces acting on the body are finite.

(j) A uniform rod is held in a vertical position with one end resting upon a perfectly rough table and when released rotates about the end in contact with the table. Find the motion.

OPTION-F

(Group Theory-II)

Paper : MAT-HE-6066

(Symbols have the usual meaning)

1. Answer ***any ten*** questions : 1×10=10

(a) Define inner automorphism induced by an element of a group.

(b) Which of the following is not True ?

(i) $U(20) \cong U(4) \oplus U(5)$

(ii) $U_4(20) \cong U(5)$

(iii) $U_5(20) \cong U(4)$

(iv) $U(20) \cong U(2) \oplus U(10)$

(c) List the elements of $Z_2 \oplus Z_3$.

(d) State whether the following statement is *true* **or** *false* :

"The additive group of integers is isomorphic to the additive group of even integers."

(e) Write the identity element of $Z_2 \oplus D_4$.

(f) Define normal subgroup of a group.

(g) State whether the following statement is true **or** false :

"Every subgroup of a cyclic group is normal."

(h) Consider the additive group of integers Z and its subgroups $\langle 3\rangle, \langle 4\rangle, \langle 5\rangle, \langle 6\rangle$ and $\langle 12\rangle$. Choose which of the following is not true :

(i) $\frac{\langle 3\rangle}{\langle 12\rangle}$ is cyclic

(ii) $\frac{\langle 4\rangle}{\langle 12\rangle}$ is cyclic

(iii) $\frac{\langle 5\rangle}{\langle 12\rangle}$ is cyclic

(iv) $\frac{\langle 6\rangle}{\langle 12\rangle}$ is cyclic

(i) How many Abelian groups (up to isomorphism) of order n are there if n is not divisible by the square of any prime ?

(j) State fundamental theorem of finite Abelian groups.

(k) State when *two* subgroups H and K of a group G are said to be conjugate.

(l) Define Sylow *p-subgroup*.

(m) In the dihedral group D_4 write the conjugate class of H.

(n) Define a simple group.

(o) Is a group of order 210 simple ?

2. Answer ***any five*** questions : 2×5=10

(a) Show that any isomorphism from a group G onto group $\overline{G}$ carries identity of G to the identity of $\overline{G}$.

(b) If T_g is an inner automorphism of a group G induced by the element g, then show that $T_{g^{-1}} = (T_g)^{-1}$.

(c) List the elements of $\frac{Z}{4Z}$.

(d) Show that the alternating group A_n of degree n is normal in S_n the symmetric group of degree n.

(e) For any group G, show that $N(a)=G \Leftrightarrow a \in Z(G)$, where $N(a)$ is the normalizer of a in G and $Z(G)$ is the center of G.

(f) Show that there is no simple group of order 216.

(g) State Sylow's second theorem.

(h) Find the number of Sylow's *p-subgroups* of a group G of order $p^n m$, where p is a prime that does not divide m.

3. Answer **any four** questions : $5\times4=20$

(a) Suppose that ϕ is an isomorphism from a group G onto the group $\overline{G}$. If $\overline{K}$ is a subgroup of $\overline{G}$, then prove that $\phi^{-1}[\overline{K}]=\{g \in G : \phi(g) \in \overline{K}\}$ is a subgroup of G.

(b) If G is a group and $g \in G$, then show that the map $T_g : G \to G$ defined by $T_g(x)=gxg^{-1}, \forall\ g \in G$ is an automorphism.

(c) For any group G, show that $\frac{G}{Z(G)}$ is isomorphic to $Inn(G)$.

(d) Show that, if a group G is internal direct product of its subgroups $H_1, H_2, \ldots H_n$, then $H_i \cap H_j = \{e\}$ for $i \neq j$ with $1 \leq i \leq n, 1 \leq j \leq n$.

(e) Let G be a finite Abelian group of order $p^n m$, where p is a prime that does not divide the positive integer m, then show that $G = H \times K$ where

$H = \{x \in G : x^{p^n} = e\}$ and

$K = \{x \in G : x^m = e\}$

(f) Show that the conjugate relation on a group G is an equivalence relation.

(g) Prove that for any group G, $cl(a)=\{a\} \Leftrightarrow a \in Z(G)$, where $cl(a)$ is the conjugate class of a in G and $Z(G)$ is the center of G.

(h) Let G be a group and H be a subgroup of G. Let S be the group of all permutations of the left cosets of H in G. Then show that there is a homomorphism from G into S whose kernel lies in H.

4. Answer ***any four*** questions : 10×4=40

(a) State and prove Cayley's theorem. 2+8=10

(b) For any two finite cyclic groups G and H, show that $G \oplus H$ is also cyclic if and only if $|G|$ and $|H|$ are relatively prime. Hence show that $Z_2 \oplus Z_{30} \cong Z_6 \oplus Z_{10}$. Is $Z_2 \oplus Z_{30} \cong Z_{60}$? 7+2+1=10

(c) Show that if a group G is the internal direct product of two of its subgroups H and K then it is isomorphic to the external direct product of H and K.

(d) Show that every group of order p^2, where p is a prime, is isomorphic to Z_{p^2} or $Z_p \oplus Z_p$.

(e) If p^k divides order of a finite group G, where p is a prime, then show that G has at least one subgroup of order p^k.

(f) If G is a finite group and $a \in G$, then show that $|cl(a)| = \frac{|G|}{|N(a)|}$. Hence show that $|G| = |Z(G)| + \sum_{a \notin Z(G)} \frac{|G|}{|N(a)|}$. 6+4=10

(g) Show that an integer of the form $2n$, where n is an odd integer greater than 1, is not the order of a simple group.

(h) For any positive integer n, show that $Aut(Z_n)$ is isomorphic to $U(n)$.

(i) When is a group G said to be internal direct product of two of its normal subgroups ? If H and K are two groups, then show that their external direct product $H \oplus K$ is also a group. 3+7=10

(j) (i) Let G be a finite group and $a \in G$, then show that $|cl(a)| = |G : C(a)|$

(ii) Show that an Abelian group of order pq, where p and q are distinct primes, is cyclic. 7+3=10

OPTION-G

(Mathematical Finance)

Paper : MAT-HE-6076

1. Answer ***any ten*** parts : 1×10=10

(a) "An interest rate in a particular situation defines the amount of money a borrower promises to pay the lender."

(This statement is True ***or*** *False)*

(b) What do you mean by 'treasury rates'?

(c) Write the full form of LIBOR.

(d) "Suppose a 5-year zero rate with continuous compounding is quoted as 5% per annum." Write it mathematically.

(e) Fill up the blank space :

A relatively simple derivative is a _____.

(f) Does bank participate in OTC market?

(g) Write the names of main types of traders executing trades.

(h) What do you mean by LIBID?

(i) Define investment asset.

(j) What do you mean by 'fixed-for-floating swap' ?

(k) What is the effect of an unexpected cash dividend on a call option price ?

(l) What is meant by the 'delta' of a stock option ?

(m) What is hedging ?

(n) Write the difference between simple interest and compond interest.

(o) Let C and P be the prices of a European call and a European put, both with a strike price of K and both defined on the same stock with price S. Write the relation among C, P, K and S.

2. Answer ***any five*** parts : 2×5=10

(a) A bank quotes an interest rate of 12% per annum with quarterly compounding. What is the equivalent rate with continuous compounding ?

(b) An investor receives Rs. 1,100 in one year in return for an investment of Rs. 1000 new. Find the percentage of return per annum with annual componding.

(c) Write the names of *three* broad categories of traders.

(d) What is the difference between a long forward position and a short forward position ?

(e) What is meant by 'index arbitrage' ?

(f) What is the difference between the forward price and the value of a forward contract ?

(g) Explain why the arguments leading to put-call parity for European options cannot be used to give similar result for American options.

(h) What is meant by the 'Gamma' of a stock option ?

3. Answer ***any four*** parts : 5×4=20

(a) A stock price is currently Rs. 25. It is known that at the end of 3 months it will be either Rs. 25 or Rs. 23. If the risk free interest rate is 9% with continuous compounding, what is the value of the stock 3 months hence ?

(b) Explain the principle of risk-neutral valuation.

(c) A debt of $ 25,000 is to be amortized over 7 years at 7% interest per annum compounded annually. What value of monthly payments will achieve this ?

(d) A loan of Rs. 10,000 has been issued for 6 years. Compute the amount to be repaid by the borrower to the lender if simple interest is changed @ 5% per year.

(e) What is the difference between buying a put option and writing a call option ? Draw their pay off diagrams as well.

(f) Write short notes on the following :

(i) Bond pricing

(ii) Forward rates

(g) Explain the binomial option pricing model with the help of an example.

(h) A stock index is currently 1,500. Its volatility is 18%. The risk-free rate is 4% per annum for all maturities and the dividend yield on the index is 2.5%. Calculate values of u, d and p when 6-month time step is used.

4. Answer ***any four*** parts : 10×4=40

(a) Discuss the Black-Scholes formula for European options.

(b) Explain the following terms briefly :

(i) Shorting

(ii) Spot rates

(iii) Forward rates

(iv) Short rate

(c) Consider a non-dividend paying stock with current value Rs. 250. Assuming that the risk-free rate is 6% p.a. compounding continuously, find the forward price and the initial value of a 1-year-long forward contract on the stock. After six months, the price of the stock is Rs. 267 and the risk-free interest rate is still 6%. What is the forward price and what is the value of the above forward contract ?

(d) Name the *six* factors that affect stock option prices. Explain *any two* of them.

(e) (i) Why would one invest in an asset whose beta is negative ?

(ii) Define swap and give an example of an interest rate swap.

Contd.

(f) A man deposits in a bank Rs. 16,000 at the end of each year, for 8 years. If the rate of interest is 10% per annum compounded annually, what would be the sum standing to his credit at the end of that period.

(g) A company has a Rs. 2 million portfolio with a beta of 1.2. It would like to use futures contracts on the Nifty 50 to hedge its risk. The index is currently standing at 8,400 and each contract is for delivery of Rs. 200 times the index. What is the hedge that minimizes risk ? What should the company do if it wants to reduce the beta of the portfolio to 0.6 ?

(h) Suppose that zero interest rates with continuous compounding are as follows :

Maturity (monthly)	Rate % per annum
3	8.0
6	8.2
9	8.4
12	8.5
15	8.6

Calculate forward interest rates for the 2nd, 3rd and 4th quarters.

(i) When compounded annually an interest rate is 12%. What is the rate when expressed with semi-annual compounding, quarterly compounding, monthly compounding, weekly compounding and daily compounding?

(j) Companies X and Y have been offered the following rates per annum on a Rs. 5 million 10-year investment :

	Fixed rate	Floating rate
Company X :	8.0%	LIBOR
Company Y :	8.8%	LIBOR

Company X requires a fixed rate investment; company Y requires a floating rate investment. Design a swap that will net a bank, acting as intermediary, 0.2 % per annum and will appear equally attractive to X and Y.
