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3 (Sem-1/CBCS) MAT HC 1

2022

MATHEMATICS

(Honours)

Paper : MAT-HC-1016

(Calculus)

Full Marks : 60

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer **any seven** of the following questions : 1×7=7

(a) Write down the n^{th} derivative of $\cos(5x+3)$.

(b) Write when the graph of a function f is said to have vertical tangent at a point $P(c, f(c))$.

Contd.

(c) Write down the value of $\lim_{x \rightarrow +\infty} x^n e^{-kx}$

(d) Evaluate $\int_0^{\pi/2} \sin^6 x dx$

(e) In terms of marginal revenue and marginal cost, when is the profit maximized?

(f) For what purpose the disk and washer methods are used?

(g) Parameterize the curve $y = 4x^2$

(h) When the graph of a vector function $\vec{F}(t)$ is said to be smooth?

(i) Determine the values of t for which the vector function $\vec{F}(t) = \frac{\hat{i} + 2\hat{j}}{t^2 + 1}$ is continuous?

(j) State the geometrical significance of the scalar triple product of vectors \vec{u} , \vec{v} and \vec{w} .

(k) Find $\int_0^{\pi} (t\hat{i} + 3\hat{j} - \sin t \hat{k}) dt$

(l) When a function f is said to be continuously differentiable on an interval I ?

2. Answer **any four** of the following questions :
 $2 \times 4 = 8$

(a) Evaluate $\lim_{x \rightarrow +\infty} \frac{3x^3 - 5x + 9}{5x^3 + 2x^2 - 7}$

(b) Using Leibnitz's rule obtain the n^{th} derivative of $y = x^3 e^x$.

(c) By integration find the length of the circle $r = 2 \sin \theta$.

(d) Let $\vec{F}(t) = \hat{i} + e^t \hat{j} + t^2 \hat{k}$ and
 $\vec{G}(t) = 3t^2 \hat{i} + e^{-t} \hat{j} + 2t \hat{k}$,

then find $\frac{d}{dt} \{ \vec{F}(t) \cdot \vec{G}(t) \}$.

(e) Find the tangent vector to the graph of the vector function $\vec{F}(t) = t^2 \hat{i} + 2t \hat{j} + e^t \hat{k}$ at the point $t = -1$.

(f) State Kepler's laws of motion.

(g) Find the volume generated by revolving about OX, the area bounded by $y = x^3$ between $x = 0$ and $x = 2$.

(h) Find the length of the polar curve $r = e^{3\theta}$, $0 \leq \theta \leq \pi/2$.

3. Answer **any three** of the following :
 $5 \times 3 = 15$

(a) Find the constants a and b that guarantee that the graph of the function defined by $f(x) = \frac{ax+5}{3-bx}$.

will have a vertical asymptote at $x = 5$ and a horizontal asymptote at $y = 3$.

(b) Evaluate :

$2+3=5$

(i) $\lim_{x \rightarrow \pi/2^-} (x - \pi/2) \tan x$

(ii) $\lim_{x \rightarrow +\infty} \left(1 + \frac{1}{2x} \right)^{3x}$

(c) If $I_n = \int_0^{\pi/4} \tan^n \theta d\theta$, then prove that
 $n(I_{n+1} + I_{n-1}) = 1$.

Hence evaluate $\int_0^{\pi/4} \tan^3 \theta d\theta$. $3+2=5$

(d) A firm determines that x units of its product can be sold daily at p rupees per unit where $x = 1000 - p$. The cost of producing x units per day is $C(x) = 3000 + 20x$.

Find the revenue function $R(x)$.

Find the profit function $P(x)$.

Assuming that the production capacity is at most 500 units per day, determine how many units the company must produce and sell each day to maximize the profit.

$1+1+3=5$

- (e) Show that a cone of radius r and height h has lateral surface area

$$S = \pi r \sqrt{r^2 + h^2}.$$

- (f) For any three vectors $\vec{a}, \vec{b}, \vec{c}$ in space, prove that $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{c} \cdot \vec{a}) \vec{b} - (\vec{b} \cdot \vec{a}) \vec{c}$.

- (g) Use cylindrical shell method to find the volume of the solid generated when the region R under $y^2 = x$ and x -axis over the interval $[0, 4]$ is revolved about the line $y = -1$.

- (h) If the non-zero vector function $\vec{F}(t)$ is differentiable and has constant length, then prove that $\vec{F}(t)$ is orthogonal to the derivative vector $\vec{F}'(t)$.

Verify this result for

$$\vec{F}(t) = \cos t \hat{i} + \sin t \hat{j} + 3\hat{k}.$$

$$3+2=5$$

4. Answer **any three** of the following :

$$10 \times 3 = 30$$

- (a) State Leibnitz's theorem. Use it to show that if $y = e^{m \cos^{-1} x}$, then

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0$$

Hence find $y_n(0)$.

$$2+5+3=10$$

- (b) Find the vertical and horizontal asymptotes (if any) of the graph of the function

$$f(x) = \frac{x^2 - x - 2}{x - 3}.$$

Find where the graph is rising, where it is falling, determine concavity, locate all critical points and points of inflection. Finally sketch the graph.

- (c) Obtain the reduction formula for $\int \sin^n x \, dx$.

Hence evaluate

(i) $\int_0^{\pi/2} \sin^n x \, dx$

(ii) $\int_0^{\pi/2} \sin^7 x \, dx$

$$5+3+2=10$$

- (d) A boy standing at the edge of a cliff throws a ball upward at an angle of 30° with an initial speed of 64 ft/s . Suppose that when the ball leaves the boy's hand, it is 48 ft above the ground at the base of the cliff.

- (i) What are the time of flight of the ball and its range?
 (ii) What are the velocity of the ball and its speed at impact?

- (iii) What is the highest point reached by the ball during its flight?

$$3+3+4=10$$

- (e) (i) Find the area of the surface generated by revolving about the x -axis the top half of the cardioid $r = 1 + \cos \theta$.

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- (ii) Using disk method find the volume generated when the region bounded by the line $y = 4 - x$ and the x -axis on the interval $0 \leq x \leq 4$ revolve about the line $x = -2$.

5

- (f) (i) Find the position vector $\vec{R}(t)$ and velocity vector $\vec{V}(t)$, given the acceleration $\vec{A}(t)$ and initial position and velocity vectors $\vec{R}(0)$ and $\vec{V}(0)$ as

$$\vec{A}(t) = t^2 \hat{i} - 2\sqrt{t} \hat{j} + e^{3t} \hat{k}$$

$$\vec{R}(0) = 2\hat{i} + \hat{j} - \hat{k}, \quad \vec{V}(0) = \hat{i} - \hat{j} - 2\hat{k}.$$

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- (ii) A particle moves along the parametric curve $x = 2t$, $y = t$. Find the position vector $\vec{R}(t)$ and velocity vector $\vec{V}(t)$ in terms of \hat{U}_r and \hat{U}_θ . 5

- (g) (i) It is projected that t years from now, the population of a certain country will be $P(t) = 50e^{0.02t}$ million.

At what rate will the population be changing with respect to time 10 years from now?

At what percentage of rate, will the population be changing with respect to time t years from now?

$$3+3=6$$

- (ii) Find the length of the curve defined by $9x^2 = 4y^3$ between the points $(0, 0)$ and $(2\sqrt{3}, 3)$. 4

- (h) A object moving along a smooth curve has velocity \vec{v} given by $\vec{v} = \frac{ds}{dt} \hat{T}$.

Deduce the expression for acceleration

$$\text{in the form } \vec{A} = \frac{d^2s}{dt^2} \hat{T} + k \left(\frac{ds}{dt} \right)^2 \hat{N}$$

where s is the arc length along the trajectory and k is the curvature. For an object moving along a helix with position vector $\vec{R}(t) = (\cos t, \sin t, t)$ at any instant t , find the tangential and normal components of acceleration.

$$5+5=10$$
