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3 (Sem-1/CBCS) MAT HC 2

2022

MATHEMATICS

(Honours)

Paper : MAT-HC-1026

(Algebra)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer **any ten** : 1×10=10

(a) Find the polar representation of $z = -3i$.

(b) State De Moivre's theorem.

(c) Let $z_0 = r(\cos t^* + i \sin t^*)$ be a complex number with $r > 0$ and $t^* \in [0, 2\pi)$. Write down the formula for n distinct n^{th} roots of z_0 .

Contd.

- (d) Identify the quantifier, set of context and property in the statement, "Every student in this classroom is at least 5ft tall."
- (e) Define implication. Give an example.
- (f) Prove by contradiction "There is no greatest integer".
- (g) Let A and B be two sets, write when $A \times B = \phi$. Justify your answer.
- (h) What is domain and range for the function $f(x) = \tan x$.
- (i) What are the options about the solutions of a system of linear equations?
- (j) Determine h such that the matrix $\begin{bmatrix} 2 & 3 & h \\ 6 & 9 & 5 \end{bmatrix}$ is the augmented matrix of a consistent linear system.
- (k) State True **or** False with justification :
"Whenever a system has free variables the solution set is infinite."

- (l) Write down the system of equations that is equivalent to the vector equation

$$x_1 \begin{bmatrix} -2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 8 \\ 5 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

- (m) Define Pivot positions in a matrix.
- (n) Prove $\vec{U} + \vec{V} = \vec{V} + \vec{U}$ for any \vec{U}, \vec{V} in \mathbb{R}^n .
- (o) Write the system of equation as a matrix equation

$$\begin{aligned} 3x_1 + x_2 - 5x_3 &= 9 \\ x_2 + 4x_3 &= 0 \end{aligned}$$

(p) Given, $A = \begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix}$ $\vec{x} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$

Compute $\vec{x}^T A^T$ and $A^T \vec{x}^T$.

- (q) A is an $n \times n$ matrix. Prove statement (i) \Rightarrow statement (ii).
- (i) A is an invertible matrix
- (ii) \exists a $n \times n$ matrix C s.t. $CA = I$

(r) A is an $n \times n$ matrix

Fill in the blank :

If two rows of A are interchanged to produce B, then $\det B = \underline{\hspace{2cm}}$.

2. Answer **any five** :

$$2 \times 5 = 10$$

(a) If $z_1 = 1 - i$ and $z_2 = \sqrt{3} + i$. Express $z_1 z_2$ in polar form.

(b) Write the 'converse' and 'contrapositive' of the following statement :

"For real numbers x and y , if xy is an irrational number then either x is irrational or y is irrational."

(c) Why may we use the contrapositive of a statement to prove the statement instead of direct proof? Justify using truth table.

(d) Produce counter examples to disapprove the following :

(i) For $x, y \in \mathbb{R}$, $|a| > |b|$ if $a > b$

(ii) For any $x \in \mathbb{R}$, $x^2 \geq x$

(e) Express the empty set as a subset of \mathbb{R} in two different ways.

(f) Express \mathbb{N} as the union of an infinite number of finite sets I_n indexed by $n \in \mathbb{N}$.

(g) Give an example of a relation that is not reflexive, not transitive but is symmetric.

(h) State True **or** False with justification :
An example of a linear combination of vectors \vec{v}_1 and \vec{v}_2 is $\frac{1}{2}\vec{v}_1$.

(i) Prove that the following vectors are linearly dependent

$$\vec{v}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \text{ and } \vec{v}_2 = \begin{bmatrix} 6 \\ 2 \end{bmatrix}.$$

(j) Evaluate the determinant by using row reduction to Echelon form

$$\begin{vmatrix} 1 & 5 & -6 \\ -1 & -4 & 4 \\ -2 & -7 & 9 \end{vmatrix}$$

3. Answer **any four** :

5×4=20

(a) Compute $z = (1 + i\sqrt{3})^n + (1 - i\sqrt{3})^n$.

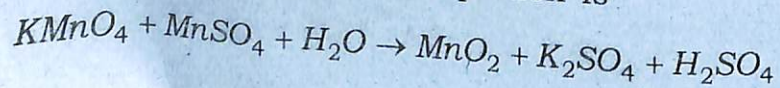
(b) Prove that the power set of a set with n elements has 2^n elements. Write down the power set of $S = \{a, b\}$.

(c) Prove that the equivalence classes of an equivalence relation on a set X induces a partition of X .

(d) Prove $(1+x)^n \geq 1+nx$ for $x \in \mathbb{R}$ such that $x > -1$ and for each $n \in \mathbb{N}$. Give the name of this inequality.

(e) Balance the chemical equation using vector equation approach the following reaction between potassium permanganate ($KMnO_4$) and manganese sulfate ($MnSO_4$) in water produces manganese dioxide, potassium sulfate and sulfuric acid.

The unbalanced equation is



(f) Find the value of h for which the set of vectors is linearly dependent

$$\begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix}, \begin{bmatrix} -6 \\ 7 \\ -3 \end{bmatrix}, \begin{bmatrix} 8 \\ h \\ 4 \end{bmatrix}$$

(g) Let A be an $m \times n$ matrix. Prove that the following statements are logically equivalent.

(i) For each $b \in \mathbb{R}^m$, the equation $A\bar{x} = \bar{b}$ has a solution.

(ii) Each $b \in \mathbb{R}^m$ is a linear combination of the columns of A .

(iii) The columns of A span \mathbb{R}^m .

(iv) A has a pivot position in every row.

(h) Use Cramer's rule to compute the solutions to the system

$$\begin{aligned} 2x_1 + x_2 &= 7 \\ -3x_1 + x_3 &= -8 \\ x_2 + 2x_3 &= -3 \end{aligned}$$

4. Answer **any four** : 10×4=40

(a) (i) Prove
$$\prod_{\substack{1 \leq k \leq n-1 \\ \gcd(k, n)=1}} \sin \frac{k\pi}{n} = \frac{1}{2^{\phi(n)}}$$

whenever n is not a power of a prime. 5

(ii) Solve the equation

$$z^7 - 2iz^4 - iz^3 - 2 = 0 \quad 5$$

(b) For any three sets A , B and C , show that

(i) $(A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A)$ 5

(ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ 5

(c) Define graph of a function verify that the set $\{(x, y) \in \mathbb{R}^2 : x = |y|\}$ is not the graph of any function. Consider the function $f: \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = ax^2 + bx + c, a \neq 0$. Show that the function is neither one-one nor onto. 2+2+6=10

(d) Let $X = \mathbb{R}$ and let

$R = \{(x, y) \in \mathbb{R}^2 : xy = 0\}$. When $x \in \mathbb{R}$ is related to $y \in \mathbb{R}$? Define reflexive, symmetric, antisymmetric and transitive relation with examples. 2+2+2+2+2=10

(e) If $A \subseteq N$, what is the least element of A ? State and prove Division Algorithm. 2+1+7=10

(f) (i) Solve the system : 5

$$x_1 - 3x_2 + 4x_3 = -4$$

$$3x_1 - 7x_2 + 7x_3 = -8$$

$$-4x_1 + 6x_2 - x_3 = 7$$

(ii) Suppose the system 3

$$x_1 + 3x_2 = f$$

$$cx_1 + dx_2 = g$$

is consistent for all possible values of f and g , what can you say about the co-efficients c and d . Justify.

(iii) Suppose a 3×5 co-efficient matrix for a system has three pivot columns. Is the system consistent? Justify. 2

(g) (i) If $\vec{U} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ $\vec{V} = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$

Display $\vec{U}, \vec{V}, \vec{U} - \vec{V}$ using arrows on an xy graph. 3

(ii) List five vectors in the span $\{\vec{v}_1, \vec{v}_2\}$

$$\vec{v}_1 = \begin{bmatrix} 7 \\ 1 \\ -6 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix} \quad 2$$

(iii) Let

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

Does $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ span \mathbb{R}^4 ? Justify. 5

(h) (i) Describe all solutions of $A\vec{x} = \vec{0}$ in parametric vector form

$$A = \begin{bmatrix} 1 & -4 & -2 & 0 & 3 & -5 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

5

(ii) Does $A\vec{x} = \vec{b}$ have at least one solution for every possible \vec{b} if A is a 3×2 matrix with two pivot positions? 2

(iii) Prove that if a set contains more vectors than the number of entries in each vector then the set is linearly dependent. 3

(i) (i) Define linear transformation. Give an example. 2

(ii) Let $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation and let A be the standard matrix for T . Then prove T maps \mathbb{R}^n onto \mathbb{R}^m if and only if the columns of A span \mathbb{R}^m . 3

(iii) Find the standard matrix of the linear transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ which is a horizontal shear transformation that leaves e_1 unchanged and maps e_2 into $e_2 + 3e_1$. 3

(iv) Show that T is a linear transformation by finding a matrix that implements the mapping

$$T(x_1, x_2, x_3) = (x_1 - 5x_2 + 4x_3, x_2 - 6x_3)$$

2

- (j) (i) Find the inverse of the matrix A (if it exists) by performing suitable row operations on the augmented matrix $[A : I]$ where

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 4 & -7 & 3 \\ -2 & 6 & -4 \end{bmatrix}. \quad 4$$

- (ii) Find the volume of the parallelepiped with one vertex at the origin and adjacent vertices at $(1, 0, -2)$, $(1, 2, 4)$ and $(7, 1, 0)$.

3

- (iii) Let the transformation

$T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be determined by a 2×2 matrix A . Prove that if S is a parallelogram in \mathbb{R}^2 then

$$\{\text{area of } T(S)\} = |\det A| \cdot \{\text{area of } S\}$$

3