

**2022**

**MATHEMATICS**

**( Major )**

**Paper : 5.2**

**( Topology )**

**Full Marks : 60**

**Time : 3 hours**

*The figures in the margin indicate full marks  
for the questions*

**1. Answer the following questions :      1×7=7**

(a) Describe the open spheres for any discrete metric space  $(X, d)$ .

(b) Find the derived sets of the following subsets of  $\mathbb{R}$  :

$$A = ]0, 1], \quad B = \left\{ \frac{2n+1}{n} : n \in \mathbb{N} \right\}$$

$$C = \left\{ -\frac{1}{n} : n \in \mathbb{N} \right\}$$

(c) Define a Cauchy sequence in a metric space  $(X, d)$ .

(d) Define a topological space and give one example.

Contd.



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(e) Let

$$X = \{a, b, c\} \text{ and}$$

$$\mathcal{T} = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$$

is a topology on  $X$ . Find the derived set of  $A = \{a, b\}$ .

(f) Let  $\mathcal{T}$  be the topology on  $\mathbb{N}$  which consists of  $\emptyset$  and all subsets of the form  $G_m = \{m, m+1, m+2, \dots\}$ ,  $m \in \mathbb{N}$ . What are the open sets containing 4?

(g) What do you mean by a Banach space? Give one example.

2. Answer the following questions :

$$2 \times 4 = 8$$

(a) Show that every closed interval is a closed set in the usual metric on  $\mathbb{R}$ .

(b) Let  $f$  be a mapping from  $\mathbb{R}$  into  $\mathbb{R}$  defined by

$$f(x) = \begin{cases} -2 & \text{when } x < 0 \\ 2 & \text{when } x \geq 0 \end{cases}$$

Examine whether  $f$  is continuous with respect to the usual topology on  $\mathbb{R}$ .

(c) Let  $(X, \|\cdot\|)$  be a normed linear space and  $x_n \rightarrow x$  and  $y_n \rightarrow y$  in  $X$ . Show that  $x_n + y_n \rightarrow x + y$ .

(d) Prove the parallelogram law in an inner product space  $(X, \langle \cdot, \cdot \rangle)$ .

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3. Answer the following questions :  $5 \times 3 = 15$

(a) Let  $(X, d)$  be a metric space and  $A$  and  $B$  be subsets of  $X$ . Prove that—

$$(i) A \subset B \Rightarrow D(A) \subset D(B)$$

$$(ii) D(A \cup B) = D(A) \cup D(B)$$

(b) Let  $X$  be any set and  $\mathcal{T}$  be the collection of all those subsets of  $X$  whose complements are finite together with the empty set. Show that  $\mathcal{T}$  is a topology on  $X$ . What do you call this topology?

Or

Let  $(X, \mathcal{T})$  be a topological space and  $A \subset X$ . Prove that  $\bar{A} = A \cup D(A)$ .

(c) Show that  $\mathbb{R}^n$  is a normed linear space with some suitable norm.

Or

Let  $(X, \langle \cdot, \cdot \rangle)$  be an inner product space. Prove that for all  $x, y \in X$

$$4\langle x, y \rangle = \|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2$$

4. Answer the following questions :  $10 \times 3 = 30$

(a) Prove that every non-empty open set on the real line is the union of a countable class of pairwise disjoint open intervals.

Or

State and prove Cantor's intersection theorem for metric spaces.

( Turn Over )



- (b) Let  $(X, d)$  be a metric space and  $x_0 \in X$  be fixed. Show that the real-valued function  $f_{x_0}(x) = d(x, x_0)$ ,  $x \in X$  is continuous. Is it uniformly continuous? Let  $(Y, P)$  be another metric space and  $f: X \rightarrow Y$  be a mapping. Prove that  $f$  is continuous if and only if the inverse image of every open set in  $Y$  is an open set in  $X$ . 2+1+7

Or

Let  $X$  be a metric space and  $Y$  be a complete metric space. Let  $A$  be a dense subspace of  $X$ . If  $f: A \rightarrow Y$  is uniformly continuous, then prove that  $f$  can be extended uniquely to a uniformly continuous mapping  $g: X \rightarrow Y$ .

- (c) Prove that a metric space is compact if and only if it is complete and totally bounded.

Or

Let  $\{A_\lambda: \lambda \in \Lambda\}$  be a family of connected subsets of a space  $X$  such that

$$\bigcap_{\lambda \in \Lambda} A_\lambda \neq \emptyset$$

Prove that  $\bigcup_{\lambda \in \Lambda} A_\lambda$  is a connected set in  $X$ .

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