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3 (Sem-6/CBCS) MAT HC 1 (N/O)

2023

MATHEMATICS

(Honours Core)

Paper : MAT-HC-6016

(New Syllabus/Old Syllabus)

Full Marks : 80/60

Time : Three hours

***The figures in the margin indicate
full marks for the questions.***

New Syllabus

Full Marks : 80

(Riemann Integration and Metric Spaces)

1. Answer the following as directed :

$1 \times 10 = 10$

- (a) Define the discrete metric d on a non-empty set X .

Contd.

(b) Let F_1 and F_2 be two subsets of a metric space (X, d) . Then

$$(i) \quad \overline{F_1 \cup F_2} = \overline{F_1} \cap \overline{F_2}$$

$$(ii) \quad \overline{F_1 \cup F_2} = \overline{F_1} \cup \overline{F_2}$$

$$(iii) \quad \overline{F_1 \cap F_2} = \overline{F_1} \cap \overline{F_2}$$

$$(iv) \quad \overline{F_1 \cap F_2} = \overline{F_1} \cup \overline{F_2}$$

(Choose the correct option)

(c) Let (X, d) be a metric space and $A \subset X$. Then

(i) $\text{Int } A$ is the largest open set contained in A .

(ii) $\text{Int } A$ is the largest open set containing A .

(iii) $\text{Int } A$ is the intersection of all open sets contained in A .

(iv) $\text{Int } A = A$

(Choose the correct option)

(d) Let (X, d) be a disconnected metric space.

We have the statements :

I. There exists two non-empty disjoint subsets A and B , both open in X , such that $X = A \cup B$.

II. There exists two non-empty disjoint subsets A and B , both closed in X , such that $X = A \cup B$.

(i) Only I is true

(ii) Only II is true

(iii) Both I and II are true

(iv) None of I and II is true

(Choose the correct option)

(e) Find the limit points of the set of rational numbers Q in the usual metric R_u .

(f) In a metric space, the intersection of infinite number of open sets need not be open. Justify it with an example.

(g) Define a mapping $f : X \rightarrow Y$, so that the metric spaces $X = [0, 1]$ and $Y = [0, 2]$ with usual absolute value metric are homeomorphic.

- (h) Define Riemann sum of f for the tagged partition (P, t) .
- (i) State the first fundamental theorem of calculus.
- (j) Examine the existence of improper Riemann integral

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$$

2. Answer the following questions : $2 \times 5 = 10$

- (a) Prove that in a metric space (X, d) every open ball is an open set.
- (b) Prove that the function $f : [0, 1] \rightarrow \mathbb{R}$ defined by $f(x) = x^2$ is an uniformly continuous mapping.
- (c) Let d_1 and d_2 be two metrics on a non-empty set X . Prove that they are equivalent if there exists a constant K such that

$$\frac{1}{K} d_2(x, y) \leq d_1(x, y) \leq K d_2(x, y)$$

- (d) If m is a positive integer, prove that $\overline{m+1} = m!$

- (e) Let $f(x) = x$ on $[0, 1]$.

$$\text{Let } P = \left\{ x_i = \frac{i}{4}, i = 0, \dots, 4 \right\}$$

Find $L(f, P)$ and $U(f, P)$.

3. Answer the following questions (**any four**):

$$5 \times 4 = 20$$

- (a) Let (X, d) be metric space and F be a subset of X . Prove the F is closed in X if and only if F^c is open.
- (b) Define diameter of a non-empty bounded subset of a metric space (X, d) . If A is a subset of a metric space (X, d) , then prove that $d(A) = d(\overline{A})$.

$$1 + 4 = 5$$

- (c) Let (X, d) be a metric space. Then prove that the following statements are equivalent :
- (i) (X, d) is disconnected.
- (ii) There exists two non-empty disjoint subsets A and B , both open in X , such that $X = A \cup B$.

- (d) Let $f, g : [a, b] \rightarrow \mathbb{R}$ be integrable functions. Then prove that $f + g$ is integrable and

$$\int_a^b (f + g)(x) dx = \int_a^b f(x) dx + \int_a^b g(x) dx$$

- (e) Discuss the convergence of the integral $\int_1^\infty \frac{1}{x^p} dx$ for various values of p .

- (f) Consider $f : [0, 1] \rightarrow \mathbb{R}$ defined by $f(x) = x^2$. Prove that f is integrable.

4. Answer the following questions : $10 \times 4 = 40$

- (a) (i) Let X be the set of all bounded sequences of numbers $\{x_i\}_{i \geq 1}$ such that $\sup_i |x_i| < \infty$.

For $x = \{x_i\}_{i \geq 1}$ and $y = \{y_i\}_{i \geq 1}$ in X define $d(x, y) = \sup_i |x_i - y_i|$.

Prove that d is a metric on X . 5

- (ii) Prove that a convergent sequence in a metric space is a Cauchy sequence. Is the converse true? Justify with an example. $4 + 1 = 5$

Or

- (a) (i) Show that $d(x, y) = \sqrt{|x - y|}$ defines a metric on the set of reals. 4

- (ii) Show that the metric space $X = \mathbb{R}^n$ with the metric given by $d_p(x, y) = \left(\sum |x_i - y_i|^p \right)^{1/p}$, $p \geq 1$ where $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$ are in \mathbb{R}^n is a complete metric space. 6

- (b) (i) Let (X, d_X) and (Y, d_Y) be two metric spaces and $f : X \rightarrow Y$. If f is continuous on X , prove the following : $3 + 3 = 6$

- (i) $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$ for all subsets of B of Y

- (ii) $f(\overline{A}) \subseteq \overline{f(A)}$ for all subsets A of X

- (ii) Let (X, d_X) and (Y, d_Y) be two metric spaces and $f : X \rightarrow Y$ be uniformly continuous. Prove that if $\{x_n\}_{n \geq 1}$ is a Cauchy sequence in X , then $\{f(x_n)\}_{n \geq 1}$ is a Cauchy sequence in Y . 4

Or

(b) Define fixed point of a mapping $T: X \rightarrow X$. Let $T: X \rightarrow X$ be a contraction of the complete metric space (X, d) . Prove that T has a unique fixed point. 2+8=10

(c) (i) Prove that if the metric space (X, d) is disconnected, then there exists a continuous mapping of (X, d) onto the discrete two element space (X_0, d_0) . 5

(ii) Let (X, d) be a metric space and A^0, B^0 are interiors of the subsets A and B respectively. Prove that

$$(A \cap B)^0 = A^0 \cap B^0;$$

$$(A \cup B)^0 \supseteq A^0 \cup B^0. \quad 5$$

Or

(c) (i) When is a non-empty subset Y of a metric space (X, d) said to be connected? Let (X, d_X) be a connected metric space and $f: (X, d_X) \rightarrow (Y, d_Y)$ be a continuous mapping. Prove that the space $f(X)$ with the metric induced from Y is connected. 5

(ii) Let (X, d) be a metric space and $Y \subseteq X$. If X is separable then prove that Y with the induced metric is also separable. 5

(d) (i) If f is Riemann integrable on $[a, b]$ then prove that it is bounded on $[a, b]$. 5

(ii) When is an improper Riemann integral said to exist? Show that the improper integral of $f(x) = |x|^{-1/2}$ exists on $[-1, 1]$ and its value is 4. 1+4=5

Or

(d) (i) Let $f: [a, b] \rightarrow \mathbb{R}$ be integrable. Then prove that the indefinite integral $F(x) = \int_a^x f(t) dt$ is continuous on $[a, b]$.

Further prove that if f is continuous at $x \in [a, b]$, then F is differentiable at x and $F'(x) = f(x)$. 3+3=6

(ii) Evaluate

$$\lim_{x \rightarrow \infty} \frac{\sqrt{1} + \sqrt{2} + \dots + \sqrt{n}}{\sqrt{n^3}} = \frac{2}{3} \quad 4$$

2. Answer the following questions : $2 \times 4 = 8$

(a) Show that $\lim_{z \rightarrow \infty} \frac{1+z^2}{z-1} = \infty$

(b) If $f(z) = e^x \cdot e^{iy} = e^z$ where $z = x + iy$, show that $f'(z) = e^x \cos y + i e^x \sin y$.

(c) Show that $\int_C f(z) dz = 0$ when the contour C is the unit circle $|z| = 1$ in either direction and $f(z) = \frac{z^2}{z-3}$.

(d) Show that the sequence $z_n = \frac{1}{n^3} + i$ ($n = 1, 2, 3, \dots$) converges to i .

3. Answer **any three** questions from the following : $5 \times 3 = 15$

(a) If z_1 and z_2 are complex numbers then show that $\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$

(b) Suppose a function $f(z)$ be analytic throughout a given domain D . If $|f(z)|$ is constant throughout D , then prove that $f(z)$ is constant in D .

(c) Show that the derivative of the real valued function $f(z) = |z|^2$ exists only at $z = 0$.

(d) If a function f is analytic at a given point, then prove that its derivatives of all orders are analytic there too.

(e) State Cauchy integral formula. Apply it to find $\int_C \frac{f(z)}{z+i} dz$ where $f(z) = \frac{z}{9-z^2}$ and C is the positively oriented circle $|z| = 2$.

4. Answer **either** (a) and (b) **or** (c) of the following questions : 10

(a) (i) Show that if $f(z) = \frac{i\bar{z}}{2}$ in the open disk $|z| < 1$, then

$$\lim_{z \rightarrow 1} f(z) = \frac{i}{2}$$

3

Contd.

(ii) Show that the function

$$f(z) = e^{-y} \sin x - i e^{-y} \cos x \text{ is entire.}$$

3

(b) If a function $f(z)$ is continuous and nonzero at a point z_0 , then prove that $f(z) \neq 0$ throughout some neighbourhood of that point.

4

Or

(c) Let the function

$f(z) = u(x, y) + i v(x, y)$ be defined throughout some ε neighbourhood of a point $z_0 = x_0 + i y_0$, and suppose that

(i) the first order partial derivatives of the functions u and v with respect to x and y exist everywhere in the neighbourhood;

(ii) those partial derivatives are continuous at (x_0, y_0) and satisfy the Cauchy-Riemann equations $u_x = v_y, u_y = -v_x$ at (x_0, y_0) .

Prove that $f'(z)$ exists and

$f'(z_0) = u_x + i v_x$ where the right hand side is to be evaluated at (x_0, y_0) .

10

5. Answer **either** (a) and (b) **or** (c) and (d) of the following questions : 10

(a) Find the value of $\int_C \bar{z} dz$ where C is the right-hand half $z = 2e^{i\theta} \left(-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\right)$ of the circle $|z| = 2$ from $z = -2i$ to $z = 2i$. 5

(b) Let C be the arc of the circle $|z| = 2$ from $z = 2$ to $z = 2i$ that lies in the 1st quadrant. Show that

$$\left| \int_C \frac{z+4}{z^3-1} dz \right| \leq \frac{6\pi}{7} \quad 5$$

Or

(c) State Liouville's theorem. 1

(d) Prove that any polynomial $p(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n$ ($a_n \neq 0$) of degree n ($n \geq 1$) has at least one zero. 9

Contd.

6. Answer **either** (a) and (b) **or** (c) and (d) of the following questions: 10

(a) Suppose that

$$z_n = x_n + iy_n \quad (n = 1, 2, 3 \dots) \text{ and}$$

$$S = X + iY. \text{ Prove that}$$

$$\sum_{n=1}^{\infty} z_n = S \text{ if and only if}$$

$$\sum_{n=1}^{\infty} x_n = X \text{ and } \sum_{n=1}^{\infty} y_n = Y. \quad 5$$

- (b) Find the Maclaurin series for the entire function $f(z) = \sin z$. 5

Or

- (c) Define absolutely convergent series. Prove that the absolute convergence of a series of complex numbers implies the convergence of the series. $1+3=4$

- (d) Find the Maclaurin series for the entire function $f(z) = \cos z$. 6