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3 (Sem–2/CBCS) MAT HC 1

2023

MATHEMATICS

(Honours Core)

Paper : MAT–HC–2016

(Real Analysis)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer the following questions as directed : $1\times10=10$

   (a) Give an example of a set which is not bounded below.

   (b) Write the completeness property of $\mathbb{R}$.

   (c) If $S=\left\{\frac{1}{n} : n\in\mathbb{N}\right\}$, then what will be $\inf S$?

(d) The unit interval $[0,1]$ in $\mathbb{R}$ is not countable.

(State whether True or False)

(e) Define a convergent sequence of real numbers.

(f) What is the limit of the sequence. $\{x_n\}$, where $x_n = \dfrac{5n+2}{n+1}$, $n \in \mathbb{N}$?

(g) A bounded monotone sequence of real numbers is convergent.

(State whether True or False)

(h) What is the value of *r* if the geometric series $\sum_{n=0}^{\infty} r^n$ is convergent?

(i) The series $1+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}+\frac{1}{\lfloor 4}+\ldots$ is not convergent.

(State whether True or False)

(j) If $\sum_{n=1}^{\infty} u_n$ is a positive term series such that $\lim_{n\to\infty} (u_n)^{1/n} = l$, then the series converges, if

(i) $l < 1$

(ii) $0 < l < 2$

(iii) $l > 1$

(iv) $1 \le l < 2$

(Choose the correct option)

2. Answer the following questions : 2×5=10

(a) Find the supremum of the set
$$S = \{x \in \mathbb{R} : x^2 - 3x + 2 < 0\}.$$

(b) If (x_n) and (y_n) are convergent sequences of real numbers and $x_n \le y_n \; \forall \, n \in \mathbb{N}$, then show that
$$\lim_{n\to\infty} x_n \le \lim_{n\to\infty} y_n.$$

(c) Show that the sequence $((-1)^n)$ is divergent.

(d) Define absolutely convergent series and give an example.

(e) Show that the series $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$ is convergent.

3. Answer ***any four*** questions : 5×4=20

(a) Prove that if $x \in \mathbb{R}$, then there exists $n_x \in \mathbb{N}$ such that $x \le n_x$.

(b) If x and y are real numbers with $x < y$, then show that there exists an irrational number z such that $x < z < y$.

(c) Show that if a sequence (x_n) of real numbers converges to a real number x, then any subsequence of (x_n) also converges to x.

(d) Show that the sequence

$$\left((-1)^n + \frac{1}{n}\right), \quad n \in \mathbb{N} \text{ is not a Cauchy}$$

sequence.

(e) Using ratio test establish the convergence or divergence of the series whose nth term is $\frac{n!}{n^n}$.

(f) Let $z = (z_n)$ be a decreasing sequence of strictly positive numbers with $lim(z_n) = 0$. Prove that the alternating series $\Sigma(-1)^{n+1} z_n$ is convergent.

4. Answer the following questions : 10×4=40

(a) Prove that the set $\mathbb{R}$ of real numbers is not countable.

Or

If S is a subset of $\mathbb{R}$ that contains at least two points and has the property : if $x, y \in S$ and $x < y$, then $[x, y] \subseteq S$, then show that S is an interval.

(b) Prove that a sequence of real numbers is convergent if and only if it is a Cauchy sequence.

Or

Let (x_n) be a sequence of positive real numbers such that $L = \lim_{n \to \infty} \frac{x_{n+1}}{x_n}$ exists. If $L < 1$, then show that (x_n) converges and $\lim_{n \to \infty} x_n = 0$.

(c) (i) Show that $\lim \left(\frac{1}{n^2 + 1} \right) = 0$ 2½

(ii) Show that the sequence $\left(\frac{1}{n} \right)$ is a Cauchy sequence. 2½

(iii) Prove that every contractive sequence is a Cauchy sequence. 5

Or

State and prove the monotone subsequence theorem. 10

(d) Prove that a positive term series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $0 < p \le 1$.

Or

Show that a necessary condition for convergence of an infinite series $\sum_{n=1}^{\infty} u_n$ is that $\lim_{n \to \infty} u_n = 0$. Demonstrate by an example that this is not a sufficient condition for the convergence.
