Total number of printed pages–8

3 (Sem–4/CBCS) MAT HC 1

2023

MATHEMATICS

(Honours Core)

Paper : MAT–HC–4016

(Multivariate Calculus)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer the following questions as directed : $1\times10=10$

   (a) If $f(x, y, z) = x^2ye^{2x} + (x+y-z)^2$, then find $f(-1, 1, -1)$.

   (b) If $f(x, y) = sin^{-1}(xy)$, then find f_y at $\left(\frac{1}{2}, \frac{1}{2}\right)$.

   (c) Define open disk in R^2.

Contd.

(d) Define critical point of a function f of two variables x and y on an open set.

(e) Let f be a function of two variables x and y defined on the open disk D and $(x_0, y_0) \in D$, then state which one of the following statements is not true :

(i) f is said to have absolute extrema at (x_0, y_0) if $f(x_0, y_0) \geq f(x, y) \; \forall \; (x, y) \in D$.

(ii) f is said to have absolute extrema at (x_0, y_0) if $f(x_0, y_0) \leq f(x, y) \; \forall \; (x, y) \in D$.

(iii) $f(x_0, y_0)$ is a relative maximum if $f(x, y) > f(x_0, y_0) \; \forall \; (x, y) \in D$.

(iv) $f(x_0, y_0)$ is a relative maximum if $f(x, y) \geq f(x_0, y_0) \; \forall \; (x, y) \in D$.

(f) Find the Jacobian of the transformation from Cartesian coordinates to polar coordinates system.

(g) If $f(x, y) = kg(x, y)$ throughout a rectangular region R, then $\iint_R f(x, y)\, dA = k \iint_R g(x, y)\, dA$. State whether this statement is true or false.

(h) Define a vector field.

(i) Find $curl\, \vec{F}$ where $\vec{F}(x, y, z) = xy\hat{i} + yz\hat{j} + zx\hat{k}$.

(j) State when a vector field C is said to be conservative.

2. Answer the following questions : 2×5=10

(a) Find the domain and range of

$$f(x, y) = \frac{1}{\sqrt{9 - x^2 - y^2}}$$

(b) Find the critical point of $f(x, y) = (x-2)^2 + (y-3)^4$ and classify them.

(c) Find $\iint_R (4 - y)\, dA$ where

$R = \{(x, y) : 0 \leq x \leq 3, \; 0 \leq y \leq 4\}$.

(d) Find $div\vec{F}$, given that $\vec{F} = \vec{\nabla} f$ where $f(x, y, z) = x^2 y z^3$.

(e) Show that the function $f(x, y) = e^{-x}(\cos y - \sin y)$ is harmonic.

3. Answer ***any four*** questions : 5×4=20

(a) Find the equation of the tangent plane and the normal line at $P_0(1,-1,2)$ on the surface S given by

$$x^2y + y^2z + z^2x = 5.$$

(b) Use the method of Lagrange's multiplier to maximize $f(x, y) = 16 - x^2 - y^2$ subject to $x + 2y = 6$.

(c) Use polar coordinates to compute the area of the region D bounded above by the line $y = x$ and below by the circle $x^2 + y^2 - 2y = 0$.

(d) Evaluate $\int_C \vec{F}.\,d\vec{R}$ where

$\vec{F} = \left(y^2 - z^2\right)\hat{i} + 2yz\,\hat{j} - x^2\hat{k}$ and C is the curve defined parametrically by $x = t^2$; $y = 2t$; $z = t$ for $0 \le t \le 1$.

(e) Verify Stokes' theorem in computing the line integral

$$\int_\Gamma x^2y^3\,dx + dy + zdz$$

where Γ is the circle $x^2 + y^2 = a^2$, $z = 0$.

(f) Evaluate $\iiint_B z^2 y e^x\,dv$ where B is the box given by

$0 \le x \le 1$; $1 \le y \le 2$; $-1 \le z \le 1$

4. Answer the following questions : 10×4=40

(a) (i) If $f(x, y, z) = xyz + x^2y^3z^4$, then show that $f_{xyz} = f_{yzx} = f_{zxy}$. 5

(ii) At a certain factory, the daily output is $Q = 60K^{\frac{1}{2}}L^{\frac{1}{3}}$ units, where K denotes the capital investment (in units of Rs. 1,000) and L the size of the labour force (in worker-hours). The current capital investment is Rs. 9,00,000, and 1,000 worker-hours of labour are used each day. Use total differential to estimate the change in output that will result if capital investment is increased by Rs. 1,000 and labour is decreased by 2 worker-hours. 5

OR

(i) Find $\frac{\partial w}{\partial s}$ if $w = 4x + y^2 + z^3$,

where $x = e^{rs^2}$, $y = \log\frac{r+s}{t}$,

$z = rst^2$. 3

(ii) The square

$S = \{(x, y) : 0 \le x \le 5, 0 \le y \le 5\}$ is heated in such a way that $T(x, y) = x^2 + y^2$ is the temperature at the point $P(x, y)$. In what direction will heat flow from the point $P_0(3, 4)$?

7

(b) Find all critical points of $f(x, y) = 8x^3 - 24xy + y^3$ and use the 2^{nd} partial derivative test to classify each point as relative extremum or a saddle point.

OR

The function $f(x, y)$ and $g(x, y)$ have continuous first order partial derivatives and f has an extremum at $P_0(x_0, y_0)$ on the smooth constraint curve $g(x, y) = c$. If $\vec{\nabla} g(x_0, y_0) \neq 0$, then show that there is a number λ such that

$\vec{\nabla} f(x_0, y_0) = \lambda \vec{\nabla} g(x_0, y_0)$.

(c) Evaluate $\iiint\limits_D (x^2 + y^2 + z^2)\,dx\,dy\,dz$

where D denotes the region bounded by $x = 0$, $y = 0$, $z = 0$ and $x + y + z = a$; $a > 0$.

OR

(i) Use double integration to find the area bounded by $y = 2 - x$ and $y^2 = 4 - 2x$. 5

(ii) Find the volume of the tetrahedron bounded by the plane $2x + y + 3z = 6$ and the coordinate planes $x = 0$, $y = 0$, $z = 0$. 5

(d) State and prove Green's theorem.

2+8=10

OR

Show that the vector field

$\vec{F} = 2x\left(y^2 + z^2\right)\hat{i} + 2x^2y\,\hat{j} + 3x^2z^2\,\hat{k}$

is conservative. Also find its scalar potentials. 4+6=10
