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3 (Sem-4/CBCS) MAT HC 3

2023

MATHEMATICS

(Honours Core)

Paper : MAT-HC-4036

(Ring Theory)

Full Marks : 80

Time : Three hours

***The figures in the margin indicate
full marks for the questions.***

1. Answer the following questions : $1 \times 10 = 10$
 - (a) Give an example of an infinite noncommutative ring that does not have a unity.
 - (b) Define an integral domain.
 - (c) What is the characteristic of the ring of 2×2 matrices over integers?
 - (d) In an integral domain, if $a \neq 0$ and $ab = ac$, then prove that $b = c$.

Contd.

(e) Show that $2\mathbb{Z} \cup 3\mathbb{Z}$ is not a subring of \mathbb{Z} .

(f) Prove that the correspondence $x \rightarrow 5x$ from \mathbb{Z}_5 to \mathbb{Z}_{10} does not preserve addition.

(g) Characteristic of every field is

(i) 0

(ii) an integer

(iii) either 0 or prime

(iv) either 0 or not prime

(Choose the correct option)

(h) Which of the following is not an integral domain?

(i) $\mathbb{Z}[x]$

(ii) $\{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$

(iii) \mathbb{Z}_3

(iv) \mathbb{Z}_6

(Choose the correct option)

(i) Consider $f(x) = 2x^3 + x^2 + 2x + 2$ and $g(x) = 2x^2 + 2x + 1$ in $\mathbb{Z}_3[x]$. Then $f(x) + g(x)$ is

(i) $2x^3 + x$

(ii) $2x^2 + 3x + 3$

(iii) $x^5 + 2$

(iv) $x^5 + 2x^3 + 2$

(Choose the correct option)

(j) The polynomial $f(x) = 2x^2 + 4$ is irreducible over

(i) \mathbb{Q}

(ii) \mathbb{C}

(iii) \mathbb{Z}

(iv) None of the above

(Choose the correct option)

2. Answer the following questions : $2 \times 5 = 10$

(a) Let R be a ring. Prove that $a(-b) = (-a)b = -(ab)$, for all $a, b \in R$.

(b) Prove that the only ideals of a field are $\{0\}$ and F itself.

(c) Show that the ring of integers is an Euclidean domain.

(d) If R is a commutative ring with unity and A is an ideal of R , show that R/A is a commutative ring with unity.

(e) Let $f(x) = x^3 + 2x + 4$ and $g(x) = 3x + 2$ in $Z_5[x]$. Determine the quotient and remainder upon dividing $f(x)$ by $g(x)$.

3. Answer **any four** questions of the following:
 $5 \times 4 = 20$

(a) Prove that

$Z[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in Z\}$
 is a ring under the ordinary addition and multiplication of real numbers.

(b) (i) If I is an ideal of a ring R such that 1 belongs to I , then show that $I = R$.

(ii) Let R be a ring and $a \in R$. Show that $S = \{r \in R \mid ra = 0\}$ is an ideal of R .
 $2 + 3 = 5$

(c) Prove that the ring of integers Z is a principal ideal domain.

(d) Let ϕ be a homomorphism from a ring R to a ring S . If A is a subring of R and B is an ideal of S , prove that

(i) $\phi(A) = \{\phi(a) \mid a \in A\}$ is a subring of S .

(ii) $\phi^{-1}(B) = \{x \in R \mid \phi(x) \in B\}$ is an ideal of R .
 $2\frac{1}{2} + 2\frac{1}{2} = 5$

(e) Let F be a field, $a \in F$ and $f(x) \in F[x]$. Prove that a is a zero of $f(x)$ if and only if $x - a$ is a factor of $f(x)$.

(f) Let F be a field, I a nonzero ideal in $F[x]$, and $g(x)$ an element of $F(x)$. Show that $I = \langle g(x) \rangle$ if and only if $g(x)$ is a nonzero polynomial of minimum degree in I .

Answer **either** (a) and (b) **or** (c) and (d) of the following questions :
 $10 \times 4 = 40$

4. (a) Prove that a finite integral domain is a field. Hence show that for every prime p , Z_p , the ring of integers modulo p , is a field.
 $4 + 2 = 6$

- (b) Show that $\frac{R[x]}{\langle x^2 + 1 \rangle}$ is a field. 4

OR

- (c) Prove that every field is an integral domain. Is the converse true? Justify with an example. 2+1=3
- (d) Define prime ideal and maximal ideal of a ring. Show that $\langle x \rangle$ is a prime ideal of $Z[x]$ but not a maximal ideal of it. 2+5=7
5. (a) Let ϕ be a homomorphism from a ring R to a ring S . Prove that ϕ is an isomorphism if and only if ϕ is onto and $\ker \phi = \{r \in R \mid \phi(r) = 0\} = \{0\}$. 5
- (b) If ϕ is an isomorphism from a ring R to a ring S , then show that ϕ^{-1} is an isomorphism from S to R . 5

OR

- (c) Let R be a ring with unity e . Show that the mapping $\phi: \mathbb{Z} \rightarrow R$ given by $n \rightarrow ne$ is a ring homomorphism. 5

- (d) Define kernel of a ring homomorphism. Let ϕ be a homomorphism from a ring R to a ring S . Prove that $\ker \phi$ is an ideal at R . 1+4=5

6. (a) State and prove the second isomorphism theorem for rings. 1+7=8

- (b) Let R be a commutative ring of characteristic 2. Show that the mapping $a \rightarrow a^2$ is a ring homomorphism from R to R . 2

OR

- (c) State and prove the third isomorphism theorem for rings. 1+6=7
- (d) Prove that every ideal of a ring R is the kernel of a ring homomorphism of R . 3

7. (a) Let F be a field. If $f(x) \in F[x]$ and $\deg f(x) = 2$ or 3 , then prove that $f(x)$ is reducible over F if and only if $f(x)$ has a zero in F . 4

- (b) In a principal ideal domain prove that an element is an irreducible if and only if it is a prime. 6

OR

- (c) Let p be a prime and suppose that $f(x) \in \mathbb{Z}[x]$ with $\deg f(x) \geq 1$. Let $\overline{f(x)}$ be the polynomial in $\mathbb{Z}_p[x]$ obtained from $f(x)$ by reducing all the coefficients of $f(x)$ modulo p . If $f(x)$ is irreducible over \mathbb{Z}_p and $\deg \overline{f(x)} = \deg f(x)$, then prove that $f(x)$ is irreducible over \mathbb{Q} . 5

- (d) Let

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0 \in \mathbb{Z}[x].$$

If there is a prime p such that

$$p \nmid a_n, \quad p \mid a_{n-1}, \dots, p \mid a_0 \quad \text{and} \quad p^2 \nmid a_0,$$

then prove that $f(x)$ is irreducible over \mathbb{Q} . 5