3 (Sem–1/CBCS) MAT HC 1

2023

MATHEMATICS

(Honours Core)

Paper : MAT–HC–1016

(Calculus)

Full Marks : 60

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer the following questions : 1×7=7

   (a) Write down the nth derivative of e^{ax}.

   (b) When a function f is said to be concave up on any open interval I ?

(c) Choose the correct answer :

Profit is maximized

(i) when marginal revenue equals marginal cost

(ii) when marginal revenue is bigger than marginal cost

(iii) when marginal revenue is less than marginal cost

(d) Write a difference between Disk method and Washer method.

(e) When a vector function $\vec{F}(t)$ is said to be continuous at t_0 ?

(f) Write Kepler's second law of planetary motion.

(g) Evaluate $\int_0^1 \frac{1}{1+x^2}\,dx$

2. Answer the following questions : 2×4=8

(a) Differentiate n times the equation

$$(1+x^2)y_2 + (2x-1)\,y_1 = 0$$

(b) Evaluate

$$\lim_{x\to+\infty} \sqrt{\frac{3x-5}{x-2}}$$

(c) Parameterize the curve $r = 2\cos^3\theta$.

(d) Determine the following vectors are orthogonal *or* not :

$$\vec{u} = 3\hat{i} + 7\hat{j} - 2\hat{k}$$
$$\vec{v} = \hat{j} - \hat{k}$$

3. Answer ***any three*** of the following : 5×3=15

(a) Evaluate the following using L'Hôpital's rule

(i) $\lim_{x\to\infty} \frac{x^4}{e^4}$

(ii) $\lim_{x\to 0} \left(\frac{x-\sin x}{x^3}\right)$.

(b) A manufacturer estimate that when x units of a particular commodity are produced each month, the total cost (in dollars) will be

$$C(x)=\frac{1}{8}x^2+4x+200$$

and all units can be sold at a price of $P(x)=49-x$ dollars per unit. Determine the price that corresponds to the maximum profit.

(c) Find the area of the top half $(0\le\theta\le\pi)$ of the cardioid $r=1+\cos\theta$.

(d) Find the tangential and normal components of acceleration of an object that moves with position vector

$$\vec{R}(t)=t\hat{i}+t^2\hat{j}$$

(e) Find the volume of the parallelepiped determined by the vectors

$$\vec{u}=\hat{i}-2\hat{j}+3\hat{k}$$
$$\vec{v}=-4\hat{i}+7\hat{j}-11\hat{k}$$
$$\vec{w}=5\hat{i}+9\hat{j}-\hat{k}$$

Answer ***either*** *a* ***or*** *b* from the following questions : 10×3=30

4. *(a)* *(i)* State and prove Leibnitz's rule. 2+4=6

(ii) If $y=\tan^{-1}y$ prove that

$$(1+x^2)y_{n+1}+2xny_n+n(n-1)y_{n-1}=0$$

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(b) *(i)* Find the points of inflexion for the function

$$f(x)=3x^5-5x^3+2$$ 5

(ii) Determine whether the graph of the given function has a vertical tangent or cusp

$$f(x)=x^{\frac{2}{3}}(2x+5)$$ 5

5. *(a)* *(i)* A regular pyramid has a square base of side L and its apex located H units above the center of its base. Derive a formula for its volume V. 6

(ii) Let D be the solid region bounded by the parabola $y = x^2$ and the line $y = x$. Find the volume of the solid generated when D is revolved about the line $y = 2$. 4

(b) (i) If $\phi(x) = \int_0^{\pi/4} tan^n x\,dx$, show that

$$\phi(n) + \phi(n-2) = \frac{1}{n-1}$$

and deduce the value of $\phi(5)$. 2+3=5

(ii) Evaluate $\int \frac{sin^4 x}{cos^2 x}\,dx$ 5

6. (a) (i) Suppose an object moves along a smooth curve C with position function $\vec{R}(t) = \langle x(t), y(t), z(t)\rangle$, where $\vec{R}'(t)$ is continuous on the interval (t_1, t_2). Then show that the object has speed $\|\vec{R}'(t)\|$. 3+3=6

(ii) Position vector of a moving object is $\vec{R}(t) = \langle e^t, \sqrt{2t} + 3, e^{-t}\rangle$
Find the speed of the object at time t and compute the distance the object travels between times $t = 0$ to $t = 1$. 4

(b) (i) Prove that acceleration of an object moving with constant speed is always orthogonal to the direction of motion. 5

(ii) Find the tangential and normal components of the acceleration of an object that moves with position vector

$$\vec{R}(t) = \left(t^3, t^2, t\right).$$ 5