

Total number of printed pages-7

3 (Sem-3/CBCS) MAT HC 1

2023

MATHEMATICS

(Honours Core)

Paper : MAT-HC-3016

(Theory of Real Functions)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer the following questions: $1 \times 10 = 10$

(a) Is 0 a cluster point of $(0,1)$?

(b) "If the limit of a function f at a point C of its domain does not exist, then f diverges at C ." (Write True or False)

(c) Define $\lim_{x \rightarrow c} f(x) = \infty$, where $A \subseteq \mathbb{R}$ and $f : A \rightarrow \mathbb{R}$ and $C \in \mathbb{R}$ is a cluster point of A .

Contd.

(d) Write sequential criterion for continuity.

(e) What do you mean by an unbounded function on a set ?

(f) Let $A, B \subseteq \mathbb{R}$ and let $f: A \rightarrow \mathbb{R}$ be continuous on A and let $g: B \rightarrow \mathbb{R}$ be continuous on B . Under what condition $g \circ f: A \rightarrow \mathbb{R}$ is continuous on A ?

(g) "If a function is continuous then it is uniformly continuous."

(Write True or False)

(h) If functions f_1, f_2, \dots, f_n are differentiable at c , write the expression for $(f_1 \cdot f_2 \cdot \dots \cdot f_n)'(c)$.

(i) The function $f(x) = x$ is defined on the interval $I = [0, 1]$. Is 0 a relative maximum of f ?

(j) Define Taylor's polynomial for a function f at a point x_0 , supposing f has an n th derivative at x_0 .

2. Answer the following questions : $2 \times 5 = 10$

(a) Use $\varepsilon - \delta$ definition of limit to show

$$\text{that } \lim_{x \rightarrow 0} \frac{x^2}{|x|} = 0.$$

(b) Show that the absolute value function $f(x) = |x|$ is continuous at every point $c \in \mathbb{R}$.

(c) Give an example of a function $f: [0, 1] \rightarrow \mathbb{R}$ that is discontinuous at every point of $[0, 1]$, but $|f|$ is continuous on $[0, 1]$.

(d) "Continuity at a point is not a sufficient condition for the derivative to exist at that point." Justify your answer.

(e) Show that $\lim_{x \rightarrow 0^+} \frac{\sin x}{\sqrt{x}} = 0$.

3. Answer **any four** parts : $5 \times 4 = 20$

(a) Prove that a number $c \in \mathbb{R}$ is a cluster point of a subset A of \mathbb{R} if and only if there exists a sequence $\{a_n\}$ in A such that $\lim a_n = c$ and $a_n \neq c$ for all $n \in \mathbb{N}$.

- (b) Show that (using ε - δ definition of limit)

$$\lim_{x \rightarrow 2} \frac{x^3 - 4}{x^2 + 1} = \frac{4}{5}$$

- (c) Prove that if $I = [a, b]$ is a closed bounded interval and if $f: I \rightarrow \mathbb{R}$ is continuous on I then f is bounded on I .

- (d) Show that if f and g are uniformly continuous on a subset A of \mathbb{R} then $f+g$ is uniformly continuous on A .

- (e) Suppose that f is continuous on a closed interval $I = [a, b]$ and that f has a derivative in the open interval (a, b) . Then there exists *at least one* point c in (a, b) such that

$$f(b) - f(a) = f'(c)(b - a).$$

- (f) Let $f: I \rightarrow \mathbb{R}$ be differentiable on the interval I . Then prove that f is increasing if and only if $f'(x) \geq 0$ for all $x \in I$.

4. Answer **any four** parts : 10×4=40

- (a) Prove that a real valued function f is continuous at $c \in \mathbb{R}$ if and only if whenever every sequence $\{c_n\}$, converging to c , then corresponding sequence $\{f(c_n)\}$ converges to $f(c)$.

- (b) (i) Show that every infinite bounded subset of \mathbb{R} has *at least one* limit point. 5

- (ii) Let $A \subseteq \mathbb{R}$, let $f: A \rightarrow \mathbb{R}$ and let $c \in \mathbb{R}$ be a cluster point of A . If $a \leq f(x) \leq b$ for all $x \in A$, $x \neq c$ and if $\lim_{x \rightarrow c} f(x)$ exist then prove that $a \leq \lim_{x \rightarrow c} f \leq b$. 5

- (c) (i) Let $I = [a, b]$ be a closed bounded interval. Let $f: I \rightarrow \mathbb{R}$ be such that f is continuous. Prove that f is uniformly continuous on $[a, b]$. 5

(ii) Show that the function $f(x) = \frac{1}{x^2}$ is uniformly continuous on $A = [1, \infty)$. 5

(d) Let $I = [a, b]$ be a closed bounded interval and let $f: I \rightarrow \mathbb{R}$ be continuous on I . Then f has an absolute maximum and an absolute minimum on I .

(e) (i) Let I be a closed bounded interval and let $f: I \rightarrow \mathbb{R}$ be continuous on I . Then the set $f(I) = \{f(x) : x \in I\}$ is a closed bounded interval. 6

(ii) Let $A, B \subseteq \mathbb{R}$ and let $f: A \rightarrow \mathbb{R}$ and $g: B \rightarrow \mathbb{R}$ be functions such that $f(A) \subseteq B$. If f is continuous at a point $c \in A$ and g is continuous at $b = f(c) \in B$, then show that the composition $g \circ f: A \rightarrow \mathbb{R}$ is continuous at c . 4

(f) (i) Let $I = [a, b]$ and let $f: I \rightarrow \mathbb{R}$ be continuous on I . If $f(a) < 0 < f(b)$ or if $f(a) > 0 > f(b)$, then prove that there exists a number $c \in (a, b)$ such that $f(c) = 0$. 6

(ii) Use the definition to find the derivative of the function $f(x) = \frac{1}{\sqrt{x}}$ for $x > 0$. 4

(g) (i) State and prove Taylor's theorem. 2+5=7

(ii) Using the Mean Value theorem prove that $|\sin x - \sin y| \leq |x - y|$ for all x, y in \mathbb{R} . 3

(h) (i) Show that

$$1 - \frac{1}{2}x^2 \leq \cos x$$

for all $x \in \mathbb{R}$ 5

(ii) Evaluate $\lim_{x \rightarrow 0} \left(\frac{1}{x^2} - \frac{1}{\sin^2 x} \right)$ 5