3 (Sem–3/CBCS) MAT HC 2

2023

MATHEMATICS

(Honours Core)

Paper : MAT–HC–3026

(Group Theory–1)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer the following questions as directed : $1\times10=10$

   (a) Define order of an element of a group.

   (b) In the group Q^* of all non-zero rational numbers under multiplication, list the elements of $\left\langle \frac{1}{2} \right\rangle$.

   (c) Find elements A, B, C in D_4 such that $AB = BC$ but $A \neq C$.

(d) Define simple group.

(e) State Cauchy's theorem on finite Abelian group.

(f) State whether the following statement is true **or** false:
"If H is a subgroup of the group G and $a \in G$, then $Ha = \{ha : a \in G\}$ is also a subgroup of G."

(g) Write the order of the alternating group A_n of degree n.

(h) Give an example of an onto group homomorphism which is not an isomorphism.

(i) State whether the following statement is true **or** false :
"If the homomorphic image of a group is Abelian then the group itself is Abelian."

(j) Which of the following statement is true ?

(a) A homomorphism from a group to itself is called monomorphism.

(b) A one-to-one homomorphism is called epimorphism.

(c) An onto homomorphism is called endomorphism.

(d) None of the above

2. Answer the following questions : 2×5=10

(a) In D_3, find all elements X such that $X^3 = X$.

(b) Consider the group Z_2 under $+_2$ and Z_3 under $+_3$. List the elements of $Z_2 \oplus Z_3$ and find $|Z_2 \oplus Z_3|$.

(c) Express $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 6 & 7 & 1 & 4 & 3 & 2 \end{pmatrix}$ as product of transposition and find its order.

(d) If $\psi : G \to G'$ is a group homomorphism and e and e' be the identity elements of the group G and G' respectively then show that $\psi(e) = e'$.

(e) Show that in a group G, if the map $f : G \to G'$ defined by $f(x) = x^{-1}$, $\forall x \in G$ is a homomorphism then G is Abelian.

3. Answer ***any four*** questions : 5×4=20

(a) Let G be a group and H be a non-empty finite subset of G. Prove that H is a subgroup of G if and only if H is closed under the operation in G.

(b) If a is an element of order n in a group and k is a positive integer then prove that

$$\langle a^k \rangle = \langle a^{\gcd(n,\,k)} \rangle \text{ and}$$

$$|a^k| = \frac{n}{\gcd(n,\,k)}.$$

(c) Show that a subgroup H of a group G is a normal subgroup of G if and only if product of two right cosets of H in G is again a right coset of H in G.

(d) If a, n are two integers such that $n \geq 1$ and $\gcd(a, n) = 1$, then prove that $a^{\phi(n)} \equiv 1 (\mathrm{mod}\, n)$, where $\phi(n)$ is the Euler's phi function.

(e) Show that any finite cyclic group of order n is isomorphic to $\frac{\mathbb{Z}}{\langle n \rangle}$, where $\mathbb{Z}$ is the additive group of integers and $\langle n \rangle = \{0, n, 2n, \ldots\}$.

(f) Let $\sigma : G \to \overline{G}$ be a group homomorphism and $a, b \in G$.

(i) Show that

$$\sigma(a) = \sigma(b) \Leftrightarrow a \ker\sigma = b \ker\sigma.$$

(ii) If $\sigma(g) = g'$ then show that

$$\sigma(g') = \{x \in G : \sigma(x) = g'\} = g \ker\sigma.$$

2+3=5

Answer ***either*** (a) ***or*** (b) from the following questions : 10×4=40

4. (a) Describe the elements of D_4, the symmetries of a square. Write down a complete Cayley's table for D_4. Show that D_4 forms a group under composition of functions. Is D_4 an Abelian group? 2+3+4+1=10

(b) Prove that every subgroup of a cyclic group is cyclic. Also show that if $|\langle a \rangle| = n$, then the order of any subgroup of $\langle a \rangle$ is a divisor of n. Moreover, show that the group $\langle a \rangle$ has exactly one subgroup $\langle a^{\frac{n}{k}} \rangle$ of order k. Find the subgroup of Z_{30} which is of order 3. 4+2+3+1=10

5. *(a)* Show that every quotient group of a cyclic group is cyclic. Give example to show that converse of this statement is not true in general. Find $\frac{\mathbb{Z}}{N}$ where $\mathbb{Z}$ is the additive group of integers and $N = \{5n : n \in \mathbb{Z}\}$. 4+3+3=10.

(b) *(i)* Show that every finite group can be represented as a permutation group. 7

(ii) Let $\phi : G \to \overline{G}$ be a group homomorphism and H be a subgroup of G. If $\overline{K}$ is a normal subgroup of $\overline{G}$ then show that $\phi^{-1}[\overline{K}] = \{k \in G : \phi(k) \in \overline{K}\}$ is a normal subgroup of G. 3

6. *(a)* *(i)* State and prove Lagrange's theorem for the order of subgroup of a finite group. Is the converse true? Justify your answer. 1+5+1=7

(ii) List the elements of $\frac{\mathbb{Z}}{4\mathbb{Z}}$ and construct a Cayley's table for it. 3

(b) *(i)* Show that any two disjoint cycles commute. 5

(ii) Let G be a group and $Z(G)$ be the center of G. If $\frac{G}{Z(G)}$ is cyclic then show that G is Abelian. 5

7. *(a)* Let G be a group and H be any subgroup of G. If N is any normal subgroup of G, then show that :

(i) $H \cap N$ is a normal subgroup of H.

(ii) N is a normal subgroup of HN.

(iii) $\frac{HN}{N} \cong \frac{H}{H \cap N}$.

2+2+6=10

(b) Let $f: G \to G'$ be an onto group homomorphism and H be a subgroup of G, H' a subgroup of G'. Prove that :

(i) $f[H]$ is a subgroup of G'.

(ii) $f^{-1}[H']$ is a subgroup of G containing $K = \ker f$, where

$$f^{-1}[H'] = \{x \in G : f(x) \in H'\}.$$

(iii) There exists a one-to-one correspondence between the set of subgroups of G containing K and set of subgroups of G'.

2+3+5=10