2018

MATHEMATICS

(Major)

Paper : 3.1

(Abstract Algebra)

Full Marks : 80

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. Answer the following as directed : 1×10=10

(a) Consider the map

$$f : \langle Z, +\rangle \rightarrow \langle G, \cdot\rangle$$

where Z is the set of integers and $G = \{-1, 1\}$, defined by

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is even} \\ -1, & \text{if } x \text{ is odd} \end{cases}$$

Now state which of the following statements is true :

(i) f is an automorphism

(ii) f is not a homomorphism

(iii) f is a homomorphism but not one-one and onto

(iv) f is an onto homomorphism but not one-one

(Choose the correct option)

(b) State the condition under which a homomorphism from a group to another group is one-one.

(c) Consider the homomorphism

$$f : C \to R$$

where C and R are the additive groups of complex and real numbers respectively, defined by $f(x+iy) = x$. Then kernel of f is

(i) the real axis

(ii) the imaginary axis

(iii) the Argand plane

(iv) Both (i) and (ii)

(Choose the correct option)

(d) State whether the following statement is True or False :

"A non-zero idempotent element of a ring cannot be nilpotent."

(e) Define simple ring.

(f) State whether the following statement is True or False :

"Every field is a vector space over itself."

(g) The order of the group of automorphisms of an infinite cyclic group is

(i) one

(ii) two

(iii) infinite

(iv) None of the above

(Choose the correct option)

(h) State whether the following statement is True or False :

"Quotient ring of an integral domain is again an integral domain."

(i) If R and S are two rings, then state under what condition S is called extension of R.

(j) State fundamental theorem of ring homomorphism.

2. Answer the following questions : $2\times5=10$

(a) Give an example to show that a subset can be isomorphic to its superset.

(b) Prove that the centre $Z(R)$ of a ring R is a subring of R.

(c) Give reason why any Abelian group of order 15 is cyclic.

(d) If T_{g_1} and T_{g_2} are any two inner automorphisms of a group G, then show that $T_{g_1} = T_{g_2}$ if and only if $g_1 Z(G) = g_2 Z(G)$ where $Z(G)$ is the center of the group G.

(e) Give example (with justification) of a ring homomorphism $f : R \to R'$ such that $f(1)$ is not unity of R' where 1 is the unity of R.

3. Answer any *four* questions : 5×4=20

(a) Let G be the multiplicative group of complex numbers whose magnitude is one, i.e.,

$$G = \{z \in C : |z| = 1\}$$

Then show that $G \cong \frac{R}{Z}$. Here R is the additive group of reals and Z is the additive group of integers.

(b) If R is a commutative ring with unity and $\langle x \rangle$ is a prime ideal of the polynomial ring $R[x]$ of R, then show that R must be an integral domain.

(c) Show that intersection of two subspaces of a vector space is again a subspace. Is union of two subspaces again a subspace? Justify your answer.

(d) Let G be a non-Abelian group of order p^3, where p is a prime. Find $o(Z(G))$ and the number of conjugate classes of G.

(e) If A and B are two ideals of a ring R, then prove that their product AB is also an ideal of R.

(f) Show that the field of quotient of an integral domain D is the smallest field containing D.

4. Answer the following questions : 10×4=40

(a) Let $f : G \to G'$ be a group homomorphism with $H = \ker f$. If K' is any normal subgroup of G' and $K = f^{-1}[K']$, then show that—

(i) K is a normal subgroup of G;

(ii) $\frac{G}{K} \cong \frac{G'}{K'}$;

(iii) H is contained in K. 3+5+2=10

Or

State and prove Cayley's theorem on a finite group. Is this theorem can be extended to an infinite group? 1+7+2=10

(b) Let R be a commutative ring with unity. Prove the following : 6+4=10

(i) An ideal M of R is maximal if and only if $\frac{R}{M}$ is a field

(ii) If every ideal of R is prime, then R is a field

Or

Prove that characteristic of an integral domain is either zero or a prime number. Also, show that if R is a finite, non-zero integral domain, then $o(R) = p^n$, where p is a prime and n is a positive integer. 5+5=10

(c) Let G be a finite group and $a \in G$, then show that

$$o(cl(a)) = \frac{o(G)}{o(N(a))}$$

where $N(a)$ and $cl(a)$ are respectively the normalizer conjugate class of a in G. Deduce that

$$o(G) = o(Z(G)) + \sum_{a \notin Z(G)} \frac{o(G)}{o(N(a))}$$ 7+3=10

Or

(i) State Sylow's first and third theorems. 3

(ii) Define inner automorphism of a group G. Prove that the set of all inner automorphisms of G is a subgroup of automorphism group of G. 1+6=7

(d) Define principal ideal domain (PID). Prove that every Euclidean domain is a PID. Also show that in a PID every non-zero prime ideal is maximal. 1+5+4=10

Or

Show that an integral domain can be imbedded into a field. 10

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